Lambda Calculus: Encoding Booleans - Worksheet

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How do you encode true and false into lambda calculus?

Lets take a look at C-like syntax (have mercy on us Dennis):
`if (p) { /*p is true or false*/ A();}else {B();}`
So when it's true we do A, otherwise (false) we do B. How do we show this with lambdas? We could make $TRUE := \lambda x.\lambda y.x$ so we get A when true, then make $FALSE := \lambda x.\lambda y.y$ so we get B when false. This is how we encode true and false into lambda calculus, and it enables us no do simple logic using just lambdas.

1 Basic logical operations

1.1 If p then a else b

As said before, if then else can be defined by:

$$IFTHENELSE := \lambda p.\lambda a.\lambda b.pab \tag{1}$$

IFTHENELSE is pretty useless given that its just application.

1.2 Logical not

Logical not returns the opposite of its arguments so:

$$NOT := \lambda x.x\ FALSE\ TRUE \tag{2}$$

1.3 Logical or

Logical or returns true unless both inputs are false so in C-like syntax it's:
`if (x) {return true;}else if (y) {return true;}else {return false;}}`
This can be reduced down to:
`if (x) {return x;}else {return y;}`
So in lambda calculus this is:

$$OR := \lambda x.\lambda y.x(xy) \tag{3}$$

1.4 Logical and

Since logical and returns true only when both inputs are true, if x is true, then y determines the result, otherwise, it's going to be false anyways.

$$AND := \lambda x.\lambda y.x(yx) \tag{4}$$

1.5 An example

$$
\begin{aligned}
(OR(AND\ TRUE\ TRUE)FALSE)(NOT\ FALSE)FALSE &\rightarrow_\beta \\
(OR(TRUE(TRUE\ TRUE))FALSE)(NOT\ FALSE)FALSE &\rightarrow_\beta \\
(OR\ TRUE\ FALSE)(NOT\ FALSE)FALSE &\rightarrow_\beta \\
(TRUE(TRUE\ FALSE))(NOT\ FALSE)FALSE &\rightarrow_\beta \\
TRUE(NOT\ FALSE)FALSE &\rightarrow_\beta \\
TRUE(FALSE\ FALSE\ TRUE)FALSE &\rightarrow_\beta \\
TRUE\ TRUE\ FALSE &\rightarrow_\beta \\
TRUE &
\end{aligned} \tag{5}
$$

Question 1 (Exercise for the reader.)

Reduce the following *easy* lambda expression, of course substituting the words for their corresponding lambdas.

$$((\lambda x.\lambda y.\lambda z.yx)a\ e\ m)(((OR\ FALSE\ TRUE)(\lambda a.\lambda b.ba)\lambda x.\lambda y.xy)y\ s) \tag{6}$$